A numerical study of the effect of degenerate Hopf bifurcations on the voltage stability in power systems

Sarai Mendoza-Armenta, Claudio R. Fuerte-Esquivel, Ricardo Becerril

**Abstract**

Bifurcation theory has been employed to qualitatively explain the mechanisms of voltage stability problems occurring in electric power systems by relating the voltage collapse and voltage oscillatory problems to the appearance of saddle node and Hopf bifurcations, respectively. The results obtained by bifurcation analysis have revealed the existence of degenerate Hopf bifurcations when two parameters of the system are varied simultaneously; however, the mechanisms of voltage stability problems when the system is operating near these kinds of bifurcations have not yet been investigated. Bearing this in mind, the main scope of this paper is to assess how the voltage stability of a power system is affected by the presence of codimension-2 degenerate Hopf bifurcations. From the multiparameter bifurcation analysis, the existence of degenerate Hopf bifurcation points are identified and a trajectory stability diagram is then computed for classifying the type of degenerate Hopf bifurcations. This classification is used for establishing the voltage dynamics’ behavior when the power system is operating near these bifurcations. Finally, the application of a StatCom is proposed to eliminate sustained voltage oscillations and the appearance of degenerate Hopf bifurcations.

© 2013 Elsevier B.V. All rights reserved.

1. Introduction

Generally speaking, voltage stability refers to the ability to control the steady voltages along the transmission network within a narrow band around nominal operating voltages. This control is accomplished by adjusting key network parameters in order to keep the power system operating at a stable equilibrium point; however, as this point is continuously moving in the parameter space, it may then become unstable for some critical parameter values.

In the real-life operation of power systems, the knowledge of the parameter space region where all equilibrium points that can be reached by a continuous slow variation of system parameters are asymptotically stable, referred to as the feasibility region [1], is of paramount importance. Similarly, the boundary location of this region in the parameter space must be known in order to avoid an operation condition where an equilibrium point becomes unstable for small changes in system parameters. In this context, bifurcation theory provides a natural framework for studying the mechanisms associated with the loss of stability of equilibrium points in the vicinity of the feasibility region boundary [2]. The application of this theory has revealed that two major local bifurcations are related to a monotonic voltage collapse and voltage oscillatory instabilities, namely saddle-node and Hopf bifurcations, respectively [1]. Furthermore, it has been also demonstrated that the boundary of a feasibility region is composed of surfaces of saddle node bifurcations (SNB), Hopf bifurcations (HB) and singularity-induced bifurcations (SIB).

The topological characteristic of the feasibility region provides very important information to the power system operators in terms of how far the equilibrium point is from the feasibility boundary. The topological structure of a feasibility region was studied in [3] based on a multiparameter bifurcation analysis of the well-known chaotic 3-bus power system [4]. An extension to this study was presented in [5] by investigating the impact of the exciter voltage limit. The contribution of these studies consisted of the observation that a degenerate Hopf bifurcation (DHB) emerges when two Hopf bifurcations surfaces collide, and that an unstable region or hole appears inside the feasibility region if two Hopf bifurcation surfaces collide with each other at their extremes. Reference [6] demonstrated that a DHB occurs at the point in the parameter space where a Hopf bifurcation and homoclinic loop bifurcation collide. The effect of this bifurcation on the dynamics of state variables associated with the induction motor slip and the tap ratio of a load tap changer are illustrated through the analysis of phase portraits.

Despite that bifurcation theory has been widely applied to study the mechanisms associated with the loss of voltage stability and
2. Feasibility region of a power system

An electric power system can be represented by a set of time-independent parameterized differential equations constrained by a set of algebraic equations (DAEs), as given by (1)

\[
\begin{align*}
\dot{x} &= f(x, y, \beta) \quad f : \mathbb{R}^{p+m+mp} \rightarrow \mathbb{R}^n \\
0 &= g(x, y, \beta) \quad g : \mathbb{R}^{p+m+mp} \rightarrow \mathbb{R}^m
\end{align*}
\]

where \(x \in X \subset \mathbb{R}^p, y \in Y \subset \mathbb{R}^m\) and \(\beta \in B \subset \mathbb{R}^{mp}\) correspond to a vector of dynamic states, algebraic states and parameters, respectively.

The region \(E_Q\) in which an electric power system remains in equilibrium is characterized by a \(p\)-dimensional equilibrium manifold composed of the set of equilibrium points satisfying

\[
E_Q = \{ (x, y, \beta) \in X \times Y \times B : f(x, y, \beta) = 0, \quad g(x, y, \beta) = 0 \}
\]

The feasibility region \(\mathcal{F}_R\) of an electric power system is a subset of \(E_Q\) composed of all equilibrium points that can be reached by a continuous variation of system parameters without causing a system's instability, while the feasibility boundary \(\partial \mathcal{F}_R\) is composed of a set of equilibrium points where the power system becomes unstable. In order to determine \(\mathcal{F}_R\) and \(\partial \mathcal{F}_R\), the stability of a given equilibrium point is assessed based on the reduction of the set of DAEs (1) to a locally equivalent set of ordinary differential equations (ODEs) and the performance of an eigenvalue analysis of the Jacobian matrix associated with the resulting set of ODEs [2]. This system reduction implies that the dynamics and stability properties of the set of DAEs (1) are locally well-represented by the set of ODEs

\[
\dot{x} = f_c(x, \psi(x, \beta), \beta)
\]

and by the stability properties associated with the linearized equivalent ODE model (4), respectively:

\[
\Delta x = \left[ \frac{\partial f(\cdot)}{\partial x} - \frac{\partial f(\cdot)}{\partial y} \left( \frac{\partial g(\cdot)}{\partial y} \right)^{-1} \frac{\partial g(\cdot)}{\partial x} \right]_{\beta} \Delta x = f_{RS}\Big|_{\beta} \Delta x
\]

In this context, the equilibrium point \(E_P\) is stable if all eigenvalues of \(f_{RS}\big|_{\beta}\) have negative real parts, \(\lambda_{f_{RS}\big|_{\beta}} \subset \mathbb{C}^-,\) and the set of equilibrium points composing the feasibility region is then given by

\[
\text{OP} = \{ (x, y, \beta) \in X \times Y \times B : f(x, y, \beta) = 0, \quad g(x, y, \beta) = 0 \}
\]

The continuation techniques allow us to trace the branch of solutions from an established equilibrium solution of (2) as system parameters are varied. These techniques involve a parameterization strategy to identify the location of each equilibrium solution on the branch, a predictor step that provides an approximation to the next equilibrium solution based on a selected step length control, and a corrector step that minimizes the error associated with the approximated solution. Considering an initial stable equilibrium point obtained by solving (2) for a given set of system parameters, the first step to perform the bifurcation analysis is to solve the set of nonlinear DAEs (2) as one system parameter is varied, referred to as the bifurcation parameter, with all the other system parameters kept constant. These solutions will generate a branch of equilibrium points and periodic solutions whose display in the parameter space produces a one-parameter bifurcation diagram. The appearance of bifurcation points along the solution branch is detected by performing a stability analysis of the linearized equivalent ODE model (4), evaluated at each equilibrium point composing the solution branch and at the single bifurcation parameter \(p_B\). The corresponding conditions related to the presence of a SNB, HB or SIB are also checked. This procedure is repeated for different values of the fixed parameters held constant to obtain a set of one-parameter bifurcation diagrams. Finally, a feasibility region and its boundary are constructed by putting together this set of bifurcation diagrams, which is equivalent to having performed a multiparameter bifurcation analysis.

Instead of computing only one equilibrium solution for each set of system parameters, multiple equilibrium solutions can be computed for a given set of parameters using heuristic optimization techniques, such that the application of the principle of continuation is not necessary. Details of this application are given in [7]. A result of this application is the direct computation of the feasible region by the simultaneous variation of multiple parameters, instead of putting together a series of one-parameter bifurcation diagrams [7].

For this purpose, we are interested in the feasibility boundary segments associated with oscillatory stability problems which are closely related to Hopf bifurcations, such that a brief description of how a HB occurs is described below. On the other hand, the set of conditions that must be satisfied for an equilibrium point to belong to the manifold of SNB or SIB are reported in [1].
2.2. Hopf bifurcation conditions

The Hopf theorem [8] states that if the linearized system \( J_{KS} \mid _{\beta_0} \) has a pair of complex conjugate eigenvalues that cross the imaginary axis as \( \beta \) varies through a critical value \( \beta^*_0 \), then for a near-critical value of \( \beta \) there exist a limit cycle close to the equilibrium point. When the bifurcation parameter \( \beta_0 \) is varied, the system (3) undergoes the Hopf bifurcation at \( \beta = \beta^*_0 \) and at the equilibrium point \( E^*_p \), where \( \lambda = x, y, \beta \) = 0, if the following conditions are satisfied: (H1) The Jacobian matrix \( J_{KS} \mid _{\beta^*_0} \) possesses a pair of complex conjugate, simple eigenvalues \( \lambda(\beta^*_0) = \pm j\omega_0 \), and there is no other eigenvalue on the imaginary axis; and (H2) \( \{ d \{ \lambda(\beta) \} / d\beta \}_{\beta = \beta^*_0} \neq 0 \). This last condition implies that the simple pair of conjugate complex eigenvalues cross the imaginary axis transversally at \( \beta = \beta^*_0 \). On the other hand, the condition (H1) indicates that the equilibrium point \( E^*_p \) is non-hyperbolic, such that the local stability of the equilibrium cannot be determined from (4) alone. However, the system is hyperbolic at values of \( \beta_0 \) close enough to \( \beta^*_0 \), and therefore the linearized system (4) permits the disclosure of the local stability of \( E^*_p \).

At a Hopf bifurcation, limit cycles emerge in a vicinity of \( E^*_p \) with an initial period of \( 2\pi \lambda_{\omega_0} \). Hopf bifurcations with stable limit cycles are called supercritical Hopf bifurcations, and those with unstable limit cycles are referred to as subcritical Hopf bifurcations [8]. When two HB points are close to each other, the variation of two system parameters could provoke a collision between these two points giving rise a DHB because the transversality condition (H2) is not longer valid. The presence of DHBs creates an unstable region or unstable hole inside the feasibility region that was regarded as continuously stable, which has an important impact on the analysis of the electric power system because its operating point cannot be freely changed within the non-convex feasibility region. Once a DHB occurs, the system dynamics in the vicinity of this bifurcation can be studied through a reductive method, such as the Lyapunov-Schmidt reduction technique [9] or by numerical simulations when the system equations become very complex. The theoretical aspects associated with this technique are described below and are demonstrated numerically in the study case presented in Section 5.

3. The Lyapunov–Schmidt method

Hopf bifurcations in generic n-dimensional systems occur in essentially the same way as the corresponding bifurcations in lower dimensional systems near the critical bifurcation parameters \( \beta^*_0 \). The simplification of the n-dimensional system (3) can be achieved by applying the method of Lyapunov–Schmidt reduction [8,9] in order to attain a simpler system of algebraic equations, which essentially correspond to the one obtained by setting the time derivatives to zero in the normal form associated with the considered bifurcation. In the case of Hopf bifurcations, the Lyapunov–Schmidt method reduces (3) to a scalar equation given by

\[
G(x, \beta) = r(x^2, \beta)x, \quad r(0, 0) = 0
\]

such that the local solutions of \( G(x^2, \beta) = 0 \) with \( x > 0 \) are in correspondence with the small amplitude of periodic solutions of (3) with a period near \( 2\pi \). Note that \( G \) is an odd function in \( x \). \( G \) has \( Z_2 \)-symmetry. The concept of \( Z_2 \)-equivalent is defined as follows [8]:

**Definition.** Let \( G(x, \beta^*_0) \) and \( h(x, \beta^*_0) \) be functions with \( Z_2 \)-symmetry. We say that \( G \) and \( h \) are \( Z_2 \)-equivalent if

\[
h(x, \beta) = S(x, \beta^*_0)G(X(x, \beta), \Theta(\beta))
\]

where the triple \( (S, X, \Theta) \) is an equivalence transformation such that \( X \) is odd in \( x \) and \( S \) is even in \( x \). If this relation holds with \( \Theta(\beta) = \beta \), we say that \( G \) and \( h \) are strongly \( Z_2 \)-equivalent.

Bearing the concept of \( Z_2 \)-equivalent in mind, we can say that if (3) satisfies (H1) and (H2), then for each set of fixed parameters \( \beta_0 = (\beta_1, \beta_2, \beta_3) \), where \( \beta_0 \neq \beta_0^* \), the reduced equation \( \beta \) is \( Z_2 \)-equivalent to the normal form of a pitchfork bifurcation \( s^2 + \delta R_0 \), where \( \epsilon = sgn\sigma (0, 0) \) and \( \delta = sgn\sigma \). On the other hand, if (H2) is not satisfied \( \beta \) is \( Z_2 \)-equivalent to one of the normal forms \( s^2 + \beta^*_0 \) or \( -s^2 + \beta^*_0 \), which exhibit interesting dynamical behaviors when they are perturbed in their universal unfolding \( s^2 + \beta^*_0 \) or \( -s^2 + \beta^*_0 \) [8], whose bifurcation diagrams as a function of \( \beta_0 \) are given in Fig. 1(a) and (b), respectively. The stable and unstable solutions are indicated in these diagrams by solid and dashed lines, respectively; this notation applies for both stationary and periodic solutions.

Note that in principle obtaining formulas involving high-order derivatives of (3) that may determine the type of DHB occurring in the power system is possible. However, for the system under analysis in this paper, the derivation of these formulas become very complex; we then proceed to assess the system dynamics in the vicinity of this bifurcation numerically. Therefore, we obtain a trajectory stability diagram of the power system under study through numerical simulations, which we then compare with the bifurcation diagrams reported in Fig. 1. These study cases are detailed in the next section.

4. Study cases

The simulation results on multiparameter bifurcation analysis as well as the computation of a feasibility region and its boundary regarding nonlinear oscillations are presented in this section. The power system under consideration in this paper, shown in Fig. 2, is the one that has been employed as a benchmark system to illustrate the nonlinear dynamics observed in electric power systems due to voltage stability problems [3-5,7].

The voltage stability study through bifurcation analysis can be considered as a single problem on which the system is modeled with all dynamics at equilibrium: short-term and long-term dynamics [10]. In this context, the dynamic equations associated with

\[
\begin{align*}
E_L &< 0 \\
y_2 &< \Phi_2 \\
V_L &< 0 \\
y_1 &< \Phi_1 \\
V_i &< \delta_m \\
I &< L \\
V_{1,c} &< \delta_m \\
R_{dc} &< \delta_m \\
C_{dc} &< \delta_m \\
V_{1,c} &< \delta_m \\
R_{dc} &< \delta_m \\
C_{dc} &< \delta_m
\end{align*}
\]

**Fig. 2.** 3-Bus power system.
with the synchronous generator, induction motor, StatCom and SVC correspond to a time scale of short-term dynamics [11].

4.1. System model

The 3-bus system consists of a load which is supplied by two generators and may be viewed as an equivalent circuit for a local area of interest connected to a large network represented by one slack generator with constant voltage magnitude and phase angle $E_g\theta^\circ$. The other generator is represented by a two-axis model including a simple faster exciter loop and a field winding on $d$-axis. The generator has a constant voltage magnitude $V_t$ at its terminals and phase angle dynamics given by the swing equations (7) and (8), the transient internal voltage magnitude on the rotor circuit is given by (9) and the excitation system behavior is mathematically represented by (10)

\[
\dot{\delta}_m = \omega_y s_m , \quad s_m = \frac{\alpha_m - \omega_y}{\omega_y} \tag{7}
\]

\[
\dot{s}_m = -D s_m + P_m - (E \delta_q + E \delta_d + (x_d - x_q)x_d) \tag{8}
\]

\[
\dot{E}_g = -E_g + (x_d - x_q)x_d + E_{fL} \tag{9}
\]

\[
\dot{E}_{fL} = -E_{fL} + K_L(V_{ref} - V_t) \tag{10}
\]

The generic dynamic load model comprises a dynamic induction motor model in parallel with a constant $P_{ld} - Q_{ld}$ load, with a demanded power given by [4]

\[
P_L = P_{ld} + P_0 + p_1 \delta_L + p_2 V_L^2 + p_3 V_L \tag{11}
\]

\[
Q_L = Q_{ld} + Q_0 + q_1 \delta_L + q_2 V_L + q_3 V_L^2
\]

where both sets \{ $P_0$, $Q_0$ \} and \{ $p_1$, $p_2$, $p_3$, $q_1$, $q_2$, $q_3$ \} are constant powers and fixed values associated with the dynamic load, respectively. A capacitor with a fixed susceptance $B_c$ is also connected at the load bus to provide reactive power support when the load reactive power demand is increased.

The active and reactive powers supplied to the load are given by (12) and (13), respectively:

\[
P_{Net} = V_L V_L y_1 \cos(\delta_L - \delta_m - \phi_1) - V_L^2 y_1 \cos(\phi_1) + E_k y_2 \cos(\delta_L - \phi_2) - V_L^2 y_2 \cos(\phi_2) \tag{12}
\]

\[
Q_{Net} = V_L V_L y_1 \sin(\delta_L - \delta_m - \phi_1) + V_L^2 y_1 \sin(\phi_1) + E_k y_2 \sin(\delta_L - \phi_2) + V_L^2 y_2 \cos(\phi_2) \tag{13}
\]

Based on the StatCom equivalent circuit detailed in [12], the set of ODEs representing the StatCom dynamics in the $dq$ synchronously rotating reference frame is given by (14), while the StatCom power flow equations at the network-side terminals are given in [15]

\[
\frac{d\delta_L}{dt} = \omega_B \left( -\frac{R_L}{L_L} i_{dl} + i_{ql} + \frac{M_k}{L_L} \cos(\alpha + \delta_L)V_{dc} - \frac{V_{dc}}{L_L} \cos(\delta_L) \right) \tag{14}
\]

\[
\frac{dl_{ql}}{dt} = \omega_B \left( -\frac{R_L}{L_L} i_{ql} - i_{dl} + \frac{M_k}{L_L} \sin(\alpha + \delta_L)V_{dc} - \frac{V_{dc}}{L_L} \sin(\delta_L) \right) \tag{15}
\]

\[
\frac{dv_{dc}}{dt} = -C_{dc} \omega_B \left( M_k (\cos(\alpha + \delta_L) i_{dl} + \sin(\alpha + \delta_L) i_{ql}) + \frac{V_{dc}}{R_{dc}} \right) \tag{16}
\]

Lastly, a proportional-integral control is used to adjust the voltage response of the StatCom as depicted in Fig. 3 [12], where $V_{dc}^*$ and $Q_{dc}^*$ correspond to the reference quantities of the voltage across the dc capacitor and the reactive power injected by the controller, respectively. The set of DAE control equations (16) and (17) is obtained assuming a decoupled control approach [12], i.e. $K_p = K_q = 0$, and the control limits are represented by (18)

\[
\dot{x}_1 = k_{2p} (V_{dc}^* - V_{dc}) , \quad \dot{x}_2 = k_{2p} \Delta I_{DL} , \quad x_3 = k_{Q} \left( \frac{Q_{dc} - Q_{dc}^*}{V_{dc}} \right) \tag{16}
\]

\[
\Delta I_{DL} = k_{1p} (V_{dc}^* - V_{dc}) + x_1 , \quad \alpha_1 = k_{1Q} \left( \frac{Q_{dc} - Q_{dc}^*}{V_{dc}} \right) + x_3 , \quad M_k = k_{1p} \Delta I_{DL} + x_2 \tag{17}
\]

\[
M_k = M_{lim} \tanh \left( \frac{M_k}{M_{lim}} \right) . \quad \alpha = \alpha_{lim} \tanh \left( \frac{\alpha_1}{\alpha_{lim}} \right) \tag{18}
\]

The set of DAEs (7)–(18) represents in expanded form the general mathematical model of a power system given by (1). However, in order to perform the multiparameter bifurcation analysis one must attain the equivalent ODE model (3) by solving for the algebraic variables in terms of the state variables. Attaining the equations that represent the dynamics of the voltage at the load bus are then necessary. This goal is achieved based on the mismatch power equations associated with the balance of power that must exist at the load bus, such that the dynamics of $V_L$ and $\delta_L$ are given by

\[
\dot{\delta}_L = \frac{Q - Q_{ld} - q_1 V_L - (q_3 - R_s V_L^2)}{q_1} , \quad V_L = \frac{(P - P_{ld} - p_1 q_1 X Q - Q_{ld} - q_0 V_L - (q_3 - R_s V_L^2) - p_3 V_L)}{p_2} \tag{19}
\]

where $Q = Q_{Net} + Q_{StatCom}$ and $P = P_{Net} + P_{StatCom}$ are used when the StatCom is embedded in the network; otherwise $Q = Q_{Net}$ and $P = P_{Net}$.

4.2. Feasibility region without StatCom

When the StatCom is not embedded in the network, the vector $y = \{E_g, i_d, i_q, P_{Net}, Q_{Net}, V_L \}^{T} \in \mathbb{R}^6$ corresponds to the algebraic variables, while the vector $x = \{d_1, \delta_m, s_m, E_g, E_{fL}, V_L \}^{T} \in \mathbb{R}^6$ represents the set of state variables. The system parameters are reported in Table 1. Based on these data, a multiparameter bifurcation analysis is performed by computing a set of one-parameter bifurcation diagrams through the XPPAUT program [13]. The XPPAUT is a
flexible program that makes use of continuation methods based on predictor–corrector algorithms to trace the path of an already established stable solution around the parameter space. Once the bifurcation diagram has been obtained, this package also permits the determination of the number and types of bifurcation points contained in this diagram or solution branch. In designing the bifurcation analysis, the parameters \( P_m \) and \( T_A \) are varied to obtain a codimension-2 DHB. \( P_m \) is considered the bifurcation parameter, which is varied within the range \([0.4–1.97 \, \text{pu}]\). In addition, \( T_A \) is the auxiliary parameter setting at \( 0.030 \, \text{s} \), and the following initial conditions are selected to obtain the path of equilibrium solutions: \( \delta_m = 0.86864 \, \text{rad}, s_m = 0.0 \, \text{rad/s}, E_q^* = 1.0556 \, \text{pu}, E_d^* = 2.3684, \delta_1 = 0.1312 \, \text{rad} \) and \( V_1 = 1.012 \, \text{pu} \). The resulting one-parameter bifurcation diagram associated with \( V_1 \) is depicted in Fig. 4 where the solid and dashed lines indicate stable and unstable equilibrium points, respectively. The eigenvalue analysis of the equilibrium points composing this path of solutions indicates the appearance of two supercritical (stable) HBs at \( P_m = 0.7808 \, \text{pu} \), \( V_1 = 1.0146 \, \text{pu} \) (SHB1) and \( P_m = 0.9659 \, \text{pu} \), \( V_1 = 1.0661 \, \text{pu} \) (SHB2), respectively, a subcritical (unstable) HB at \( P_m = 1.2026 \, \text{pu} \), \( V_1 = 0.9890 \, \text{pu} \) (UBH1) and a SNB at \( P_m = 1.9608 \, \text{pu} \), \( V_1 = 0.7860 \, \text{pu} \).

The bifurcation analysis described above is repeated for values of \( T_A \) \([0.01–0.25 \, \text{s}]\) in order to obtain the multiparameter bifurcation diagram shown in Fig. 5. The examination of this diagram indicates that the system’s feasibility region is bounded by three segments connecting the same kind of bifurcation points: the segment HB1 connects the set of supercritical HB points SHB1, while the segments HB2 and HB3 are related to the set of supercritical and subcritical bifurcation points SHB2 and UBH1, respectively. The segments HB1 and HB3 coalesce at points \( P_m = 0.8723, T_A = 0.02862 \) and \( N (P_m = 0.8347, T_A = 0.2260) \) producing a bounded instability region, referred to as a hole, inside the feasibility region, resulting in the non-convex geometric structure of this region.

The collision of the two segments HB1 and HB2 gives rise to the appearance of a codimension-2 DHB at each point \( M \) and \( N \), indicating that the transversal condition \((H2)\) is no longer satisfied because the critical eigenvalues do not cross the imaginary axis. This statement can be numerically validated by computing the movement of the real part of the critical complex conjugate eigenvalue of \((4)\) as a function of the change in values of \( P_m \) and \( T_A \), as shown in Fig. 6; (a detailed analysis of the appearance of DHBs is given in [3,5]). Note that for values of \( T_A = (0.02862, 0.226) \), any selected trajectory describing the critical eigenvalue movement crosses the imaginary axis twice, giving rise to the SHB1 and SHB2 points shown in Fig. 4. Finally, the critical eigenvalue is tangent to the imaginary axis at points \( M (P_m = 0.8723, T_A = 0.02862) \) and \( N (P_m = 0.8347, T_A = 0.226) \), respectively, such that \((d |\text{Re}(\lambda(\beta_B))|/d\beta_B)_{\beta_B^*} = 0\) and, a DHB occurs at each of these points.

### 4.3. Degenerate case study of voltage stability

Owing to the fact that the conclusions of the Hopf theorem are only valid in such an extremely small neighborhood of the bifurcation parameters related to the DHB [9], directly studying the voltage stability problems in the time domain and detecting the existing connection of these trajectories with the types of DHB depicted in Fig. 1 is desirable. In this case, the system trajectories are obtained numerically, and the type of degenerate cases is analyzed by projecting the trajectories onto the two-dimensional space from the state space \( \mathbb{R}^n \). For each bifurcation parameter near the critical value \( \beta_B^* \) associated with points \( M \) and \( N \), we numerically solve the set of ODEs \((3)\) at different initial conditions of \( \mathbf{x} \) and observe each trajectory to determine whether it is a stationary or a periodic solution. Each solution is then drawn as a point in a diagram on the \( x_1 - p_B \) plane, referred to as the trajectory stability diagram, and each point corresponds to a periodic or stationary solution in the bifurcation diagrams of Fig. 1 both in the structure and in the stability [9]. This correspondence implies that there exists a one-to-one relationship between the bifurcation and trajectory stability diagrams.

The results of the trajectory stability diagrams in the vicinity of points \( M \) and \( N \) are depicted in Figs. 7 and 8, respectively, where the solid lines are stationary solutions while dashed lines are related to periodic solutions.
From Fig. 7 the stationary solutions are all stable, and periodic solutions do not occur in the vicinity of point M. Therefore, the fact that (H2) is violated implies the absence of limit cycles. Comparing Fig. 7 with Fig 1(a) we declare that this degenerate case qualitatively corresponds to the bifurcation diagram with $\beta = 0$. Hence, the time evolution of $V_L$ tends to a stable equilibrium point for initial conditions near to point M, as shown in Fig. 9 for initial conditions: $\delta_m = 0.91931$ rad, $s_m = 0.0$ rad/s, $E_q = 1.0496$ pu, $E_d = 2.4209$, $\delta_L = 0.15579$ rad, $V_L = 1.08$ pu, $P_m = 0.8817$ pu and $T_d = 0.02862$ s.

The trajectory stability diagram for the degenerate case occurring at point N includes stationary solutions and semi-stable limit cycles. The correspondence of this diagram is with the bifurcation diagram shown in Fig. 1(b) at a value of $\beta = 0$. When the initial conditions are located in the area between both types of solutions, the load voltage evolution is attracted to a stable equilibrium point as shown in Fig. 10 for the initial conditions: $\delta_m = 0.86744$ rad, $s_m = 0.0$ rad/s, $E_q = 1.0557$ pu, $E_d = 2.3675$, $\delta_L = 0.13074$ rad, $V_L = 1.013$ pu, $P_m = 0.8395$ pu and $T_d = 0.226$ s. However, note that the time required to attain a steady-state operating point is much larger than when the system is operating near the DHB located at point M. On the other hand, if the initial conditions are specified in the region bounded by limit cycles, the system dynamics are attracted to a limit cycle, as shown in Fig. 11 for the following initial conditions: $\delta_m = 0.86744$ rad, $s_m = 0.0$ rad/s, $E_q = 1.0557$ pu, $E_d = 2.3675$, $\delta_L = 0.13074$ rad, $V_L = 1.03$ pu, $P_m = 0.8395$ pu and $T_d = 0.226$ s. Note that the equilibrium points are stable before and after the DHB point N while the limit cycles are semi-stable before and after this point. Therefore, the violation of H2 implies that the system stability properties do not change in the neighborhood of point N: the limit cycles and equilibrium points preserve their stability characteristics on both sides of the DHB point N.

---

Fig. 6. Loci of the critical eigenvalue: (a) $T_d \leq 0.06$ s. (b) $T_d \geq 0.18$ s.

Fig. 7. Trajectory stability diagram close enough to the DHB point M.

Fig. 8. Trajectory stability diagram close enough to the DHB point N.

Fig. 9. Time evolution of $V_L$ near the DHB point M.

Fig. 10. Time evolution of $V_L$ near the DHB point N.
4.4. Voltage collapse scenarios near the DHBs

In this section we are interested in finding the initial conditions of \( V_l \) that produce a voltage collapse scenario for every value of \( P_m \) near the DHBs. Figs. 12 and 13 report the boundary values \( V_l^{B} \), represented by dots, where for values of \( V_l \) located at the top of both figures (\( V_l \leq V_l^{B} \) located at the bottom of both figures) a voltage collapse phenomenon occurs. Note that the initial conditions for \( V_l \) are far, from an operating condition view point, from the DHB located at point \( M \); however, it is not the case for the other DHB where initial conditions around \( V_l = 1.06 \) pu can produce a voltage collapse. By way of example, Fig. 14 shows a voltage collapse scenario near the DHB at point \( N \) for the initial conditions: \( \sigma_m = 0.90243 \) rad, \( s_m = 0.0 \) rad/s, \( E_m = 1.0515 \) pu, \( E_{lid} = 2.4031 \), \( \delta_l = 0.14756 \) rad, \( V_l = 1.065 \) pu, \( P_m = 0.8679 \) pu and \( T_A = 0.226 \) s.

![Fig. 11. Periodic behavior of \( V_l \) close enough to the DHB point \( N \).](image)

![Fig. 12. Initial conditions for voltage collapse near the DHB \( M \).](image)

![Fig. 13. Initial conditions for voltage collapse near the DHB \( N \).](image)

![Fig. 14. Voltage collapse near the DHB located at point \( N \).](image)

![Fig. 15. One-parameter bifurcation diagram with StatCom.](image)

### 4.5. Feasibility region with StatCom

The effect of the StatCom on the feasibility region topology is studied by connecting this controller at the load bus. The controller can provide adjustable reactive power support to increase the loadability margin and to prevent voltage magnitude oscillations. Hence, the control objectives for the StatCom are to provide independent reactive power support and to maintain constant \( V_{dc} \) through the modulation index \( M_k \) and phase angle \( \alpha \) of its PI-based control. The StatCom parameters have been selected as reported in Table 2.

The set of state variables is given by the vector \( x = [\delta_l, \delta_m, s_m, E_m, E_{lid}, l_{ql}, \omega_m, V_l, V_{dc}, x_1, x_2, x_3]^T \in \mathbb{R}^{12} \) while \( y = [\alpha, M_k, E_{lid}, l_{ql}, P_{Net}, P_{StatCom}, Q_{Net}, Q_{StatCom}, V_{dc}]^T \in \mathbb{R}^{10} \) is the vector of algebraic variables. Fig. 15 depicts the one-parameter bifurcation diagram obtained from a stable equilibrium point defined by the following initial conditions: \( \sigma_m = 0.8684 \) rad, \( s_m = 0.0 \) rad/s, \( E_m = 1.0556 \) pu, \( E_{lid} = 2.3684 \), \( \delta_l = 0.1312 \) rad, \( V_l = 1.012 \) pu, \( T_A = 0.05 \) s, \( l_{ql} = -0.21302 \), \( l_{ql} = -0.1521 \), \( V_{dc} = 1.95 \) pu, \( x_1 = 0.0 \) pu, \( x_2 = 0.5734 \) pu, \( x_3 = -0.001955 \) pu and \( Q_{dc} = 0.1 \) pu.

An analysis of this bifurcation diagram shows that the solution path where the HBs were originally located is now completely stable: it is now composed of stable equilibrium points. Furthermore, the appearance of a HB has been delayed, and only one subcritical HB is detected at \( P_m = 1.3348 \) pu (\( V_l = 0.9698 \) pu), which increases

<table>
<thead>
<tr>
<th>Table 2: StatCom parameters.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Variable</td>
</tr>
<tr>
<td>Value</td>
</tr>
</tbody>
</table>


the system loadability before an oscillatory phenomenon appears in the voltage magnitude at the load bus. Note that the first HB point was originally detected at \( P_m = 0.7808 \text{pu} \) (Fig. 4). The disappearance of these HBs has an important impact on the feasibility region topology because DHBs do not occur when the time constant of the excitation system \( T_a \) is also varied together with \( P_m \); the feasibility region contains no unstable region or hole. This is illustrated in the two-parameter bifurcation diagram depicted in Fig. 16.

5. Conclusions

This paper presents a comprehensive analysis of the effect of degenerate Hopf bifurcations and the StatCom on the topological structure of a feasibility region associated with the voltage stability problem. A DHB emerges when two HB points collide due to the variation of two system parameters, and the presence of two DHBs creates an unstable region inside the feasibility region that was regarded as continually stable. Trajectory stability diagrams are computed for classifying the type of the degenerate case we found in the system under analysis, and the voltage stability close enough to these bifurcations has been quantitatively analyzed through numerical simulations. We have also been demonstrated numerically that the utilization of a StatCom significantly improves the power system voltage stability by avoiding the appearance of an unstable voltage inside the feasibility region.

Acknowledgement

This work was supported by Conacyt, México, under the PhD scholarship 202024 and the research project 106198.

References